The Mass of an Isothermal Planetary Atmosphere

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The mass of a planetary atmosphere is easily found from the basic principle of *hydrostatic equilibrium*. That is, the atmosphere is not exploding or imploding, but stationary in time as regards where it keeps its mass. In that case, consider a flat geometry where only the altitude coordinate z matters. The change in pressure dP across some tiny change in altitude is caused by the weight of the air in the slab pressing down under gravity. If the air density is ρ and the acceleration due to gravity is g, then the change of pressure is

$$dP = -\rho g dz \tag{1}$$

where the negative sign reflects the fact that z is a coordinate pointing straight up, but g is an acceleration pointing straight down.

The other fundamental equation we need is the idea gas law:

$$P = \frac{\rho RT}{\mu} \tag{2}$$

where T is the temperature in Kelvin and μ is the mean molecular weight of one mole of the gas. The mean molecular weight of O_2 is 0.032 kg mol^{-1}. The gas constant R is 8.314 J K^{-1} mol^{-1}.

Differentiating the gas law gives

$$dP = \frac{d\rho RT}{\mu} + \frac{\rho R dT}{\mu} \tag{3}$$

However, we are about to assume that the temperature is constant, so that $dT = 0$ and the second term vanishes. Substituting into equation 1,

$$\frac{d\rho RT}{\mu} = -\rho g dz \tag{4}$$

or

$$\frac{d\rho}{\rho} = -\frac{\mu g dz}{RT} \tag{5}$$

which if we assume T to be constant is easy to integrate. Let height z be zero at a reference density ρ_0. We integrate

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\frac{\mu g}{RT} \int_0^z \tag{6}$$

to get

$$\ln \rho - \ln \rho_0 = -\frac{\mu g z}{RT} \tag{7}$$

or, exponentiating both sides

$$\rho = \rho_0 \exp(-\frac{\mu g z}{RT}) \tag{8}$$

If we define the *scale height* H as

$$\text{exponential scale height} = H = \frac{RT}{\mu g} \tag{9}$$

then

$$\rho = \rho_0 \exp(-\frac{z}{H}) \tag{10}$$

The scale height is an interesting quantity. If the mean molecular weight μ or the surface gravity g is smaller, the scale height increases. One has a taller atmosphere. So an atmosphere composed of hydrogen (H_2, $\mu = 0.002$ kg) would extend sixteen times higher than an O_2 atmosphere, all else being equal.

The same would be true of a lunar atmosphere compared to the terrestrial atmosphere. Since the lunar gravity is about a sixth of earth's, an atmosphere of similar composition should be about six times thicker.

To estimate the mass of a planetary atmosphere, we will need the radius of the body r and the acceleration of gravity at the surface. Total mass M should be the surface area of the sphere ($A = 4\pi r^2$) times the mass in a vertical column. We integrate

$$M = A\int_0^\infty \rho dz = A\int_0^\infty \rho_0 \; exp(-\frac{z}{H})dz = \rho_0 A\{0 - (-H)\} = \rho_0 AH \tag{11}$$

De-simplifying equation 11 back to physical quantities, we have for the mass of the atmosphere

$$M = \frac{4\pi RT\rho_0 r^2}{\mu g} \tag{12}$$

Let us make an example the moon. When substituting values into the formula, we merely need to make sure to keep all quantities in the mks system of units, and never mix in kilometers or grams. An O_2 atmosphere has $\mu = 0.032$ kg mol^{-1} and we seek an average, earthlike temperature of $T = 289$ K. The lunar radius is $r = 1,737,400$ m, and the acceleration due to gravity is $g = 1.62$ m s^{-2}. The gas constant $R = 8.314$ J K^{-1} mol^{-1}. If we make the surface pressure equal to earth's oxygen partial pressure at sea level then the surface density $\rho_0 = 21\% \times 1.225$ kg m^{-3} $= 0.257$ kg m^{-3}.

$$M_{\text{lunar\_atmos}} = \frac{4\pi RT\rho_0 r^2}{\mu g} = 0.45 \times 10^{18} \text{ kg} \tag{13}$$

This compares with the (actual, nonisothermal) earth's atmosphere mass of 5.15×10^{18} kg (the formula with $\mu = 0.0289$ kg mol^{-1}, $g = 9.80$ m s^{-2}, and $r = 6,371,000$ m yields 5.3×10^{18} kg, only 1% off).